MAT 1332: Calculus for the Life Sciences II
University of Ottawa, Winter 2008
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Practice Problems, Several variables

Question 1: Find the general expression of the level sets L_c of the function

$$f(x,y) = \sqrt{16 - 4x^2 - y^2},$$

i.e., find a formula for all the points (x,y) such that $f(x,y) = c$. Draw the level sets in the x-y-plane for $c = 4$ and $c = 0$.

Solution: We write $f(x,y) = c$ and solve for y to get

$$y = \pm\sqrt{16 - c^2 - 4x^2}.$$

Hence, for $c = 4$ we get $y = \pm\sqrt{-4x^2}$, which gives only one solution, namely the point $(0,0)$. For $c = 0$ we get $y = \pm\sqrt{16 - 4x^2}$ which is an ellipse in the x-y-plane.

Question 2: Consider the function of two variables

$$f(x,y) = 4x^4 - 5x^2y^2 + y^4.$$

(a) Find the zero level set of the function, i.e., all the points (x,y) where $f(x,y) = 0$. Graph the level sets in the x-y-plane where x, y range from -2 to 2. [Hint: use $z = y^2$, then solve the quadratic equation for z and finally replace $y = \pm\sqrt{z}$.]
(b) Find the gradient of the function.
(c) Evaluate the gradient at the point $(0,1)$ and find the linear approximation of f at this point.

Solution: Setting $f(x,y) = 0$, and $y^2 = z$ we get the equation

$$z^2 - 5x^2z + 4x^4 = 0.$$

Solving for z gives

$$z = \frac{5}{2}x^2 \pm \sqrt{\frac{25}{4}x^4 - 4x^4} = x^2\left[\frac{5}{2} \pm \sqrt{\frac{9}{4}}\right] = x^2\left[\frac{5}{2} \pm \frac{3}{2}\right].$$

Hence the solutions are $z = 4x^2$ and $z = x^2$. Replacing $y = \pm\sqrt{z}$, we get the solution

$$y = \pm 2x, \qquad y = \pm x.$$

Hence the zero set of the function are straight lines in the x-y-plane with slopes ± 1, and ± 2.

Question 3: Find the Jacobi matrix of the vector-valued function

$$\mathbf{F}(x,y) = \begin{bmatrix} x^2 - 3xy + y^2 \\ \frac{x}{x+y} \end{bmatrix}$$

and evaluate it at the point $(x,y) = (2,1)$. Find the linear approximation of the function at this point.

Solution: To find the Jacobi matrix, we find the partial derivatives of the first and the second component of $\mathbf{F}$. The matrix is

$$J = \begin{bmatrix} 2x - 3y & -3x + 2y \\ \frac{y}{(x+y)^2} & -\frac{x}{(x+y)^2} \end{bmatrix}.$$

At the point (2,1), we have $\mathbf{F} = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$ and

$$J = \begin{bmatrix} 1 & -4 \\ 1/9 & -2/9 \end{bmatrix}.$$

This gives the linear approximation as

$$\mathbf{F} \approx \begin{bmatrix} -1 \\ 2/3 \end{bmatrix} + \begin{bmatrix} 1 & -4 \\ 1/9 & -2/9 \end{bmatrix} \begin{bmatrix} x-2 \\ y-1 \end{bmatrix} = \begin{bmatrix} x - 4y + 1 \\ (x - 2y + 4)/9 \end{bmatrix}.$$